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The tensorial conservation law in general relativity

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Abstract. A general tensorial conservation law is formulated by starting from the invariance of the gravitational Lagrangian density by Rosen. Utilising this new formula, the author derives some reasonable results for the mass–energy distribution which are in accordance with the Newtonian formulae.

1. Formulation of the problem

The formulation of the energy–momentum conservation law proposed by Einstein (1916a, b) for the gravitational theory leads to results for the total energy of the closed system which are physically unacceptable, unless quasi-Galilean systems are employed. The coordinate-dependent consequence of this conservation law led to some early controversy in the literature. Nevertheless the energy–momentum law originally formulated by Einstein was generally accepted.

The purpose of this paper is to propose another expression of the conservation law by starting from the invariance of the gravitational Lagrangian density by Rosen (1940, 1963), which is invariant with respect to the arbitrary space–time transformations. We shall investigate some particular applications of this new scheme in §§ 3, 4.

2. The derivation of the general tensorial conservation

Following Rosen (1940, 1963), the Lagrangian density of the gravitational field is of the form

$$\tilde{\mathscr{L}} = (\sqrt{-g}/16\pi) g^{\mu\rho}(\Delta^{\nu}_{\mu\rho}\Delta^{\sigma}_{\nu\sigma} - \Delta^{\sigma}_{\mu\nu}\Delta^{\nu}_{\rho\sigma}) \tag{2.1}$$

with

$$\Delta^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} - \bar{\Gamma}^{\mu}_{\nu\rho}, \tag{2.2}$$

where $\Gamma^{\nu}_{\mu\rho}$ denotes Christoffel symbols of the actual Riemann space–time, and $\bar{\Gamma}^{\nu}_{\mu\rho}$ denotes Christoffel symbols of the background space–time.

Since the Lagrangian density is a scalar density, we have

$$\delta\tilde{\mathscr{L}} + (\partial/\partial\chi^{\alpha})(\xi^{\alpha}\varepsilon\tilde{\mathscr{L}}) = 0 \tag{2.3}$$

under the infinitesimal transformations of

$$
\begin{aligned}
&'\chi^{\mu} = \chi^{\mu} + \varepsilon_{(n)}\xi^{\mu}_{(n)}, \qquad n = 1, 2, 3, \ldots, N,\\
&\delta' g^{\mu\nu} = g^{\nu\alpha}\frac{\partial}{\partial\chi^{\alpha}}\varepsilon\xi^{\mu} + g^{\mu\alpha}\frac{\partial}{\partial\chi^{\alpha}}\varepsilon\xi^{\nu} - \varepsilon\xi^{\alpha}\frac{\partial}{\partial\chi^{\alpha}}g^{\mu\nu},\\
&\delta'\bar{g}^{\mu\nu} = \bar{g}^{\nu\alpha}\frac{\partial}{\partial\chi^{\alpha}}\varepsilon\xi^{\mu} + \bar{g}^{\mu\alpha}\frac{\partial}{\partial\chi^{\alpha}}\varepsilon\xi^{\nu} - \varepsilon\xi^{\alpha}\frac{\partial}{\partial\chi^{\alpha}}\bar{g}^{\mu\nu}.
\end{aligned}
\tag{2.4}
$$

By using the general formula in Moller (1959), from (2.3)–(2.4), we have

$$\Theta^{\nu}_{\mu}\xi^{\mu}_{(n)} + U^{\nu\rho}_{\mu}(\partial/\partial\chi^{\rho})\xi^{\mu}_{(n)} = 0 \tag{2.5}$$

with

$$\Theta^{\nu}_{\mu} = (\partial/\partial\chi^{\rho})U^{\nu\rho}_{\mu} \tag{2.5a}$$

and

$$
\begin{aligned}
&\tilde{U}^{\nu\rho}_{\mu} = \tfrac{1}{2}[(\partial\hat{\mathscr{L}}/\partial g^{\mu\sigma}_{,\rho})g^{\nu\sigma} - (\partial\tilde{\mathscr{L}}/\partial g^{\mu\sigma}_{,\nu})g^{\rho\sigma}] = -\tilde{U}^{\rho\nu}_{\mu},\\
&\bar{U}^{\nu\rho}_{\mu} = \tfrac{1}{2}[(\partial\hat{\mathscr{L}}/\partial\bar{g}^{\mu\sigma}_{,\rho})g^{\nu\sigma} - (\partial\hat{\mathscr{L}}/\partial\bar{g}^{\mu\sigma}_{,\nu})\bar{g}^{\rho\sigma}] = -\bar{U}^{\rho\nu}_{\mu},\\
&U^{\nu\rho}_{\mu} = \hat{U}^{\nu\rho}_{\mu} + \bar{U}^{\nu\rho}_{\mu}.
\end{aligned}
\tag{2.5b}
$$

Since $(\partial/\partial\chi^{\rho})(\partial/\partial\chi^{\nu})(\xi^{\mu}\bar{U}^{\nu\rho}_{\mu}) \equiv 0$ (see Belinfante 1955, Belinfante and Carrison 1962), we obtain from (2.5)–(2.6)

$$\bar{\nabla}_{\nu}\tilde{\Theta}^{\nu}_{(n)} = \frac{1}{\sqrt{-\bar{g}}}\frac{\partial}{\partial\chi^{\nu}}\sqrt{-\bar{g}}\,\tilde{\Theta}^{\nu}_{(n)} = \frac{1}{\sqrt{-\bar{g}}}\frac{\partial}{\partial\chi^{\nu}}\frac{\partial}{\partial\chi^{\rho}}(\xi^{\mu}_{(n)}\tilde{U}^{\nu\rho}_{\mu}) = 0 \qquad (n = 1, 2, 3, \ldots, N) \tag{2.6}$$

(the differential conservation law) with

$$\tilde{\Theta}^{\nu}_{(n)} = \bar{\nabla}_{\rho}[1 - (1/\sqrt{-\bar{g}})\xi^{\mu}_{(n)}\tilde{U}^{\nu\rho}_{\mu}], \tag{2.7}$$

$$\tilde{U}^{\nu\rho}_{\mu} = (\sqrt{-\bar{g}}/16\pi)\tilde{g}_{\mu\lambda}\bar{\nabla}_{\beta}\mathscr{T}^{\rho\nu\beta\lambda}, \tag{2.8}$$

and

$$
\begin{aligned}
&\mathscr{T}^{\rho\nu\beta\lambda} = \tilde{g}^{\rho\beta}\tilde{g}^{\lambda\nu} - \tilde{g}^{\rho\lambda}\tilde{g}^{\beta\nu},\\
&\tilde{g}^{\mu\nu} = (g/\bar{g})^{1/2}g^{\mu\nu}, \qquad \tilde{g}_{\mu\nu} = (\bar{g}/g)^{1/2}g_{\mu\nu},\\
&g = \det(g_{\mu\nu}), \qquad \bar{g} = \det(\bar{g}_{\mu\nu}),
\end{aligned}
\tag{2.9}
$$

where $\bar{\nabla}_{\beta}$ and ∇_{β} denote the convariant derivatives based on Christoffel symbols of $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively.

By applying the Gauss–Stokes theorem to a region σ on a hypersurface Σ, from (2.6)–(2.7), we get

$$I(\xi_{(n)}) = \int_{\sigma}\frac{\partial}{\partial\chi^{\rho}}\xi^{\mu}_{(n)}\hat{U}^{\nu\rho}_{\mu}\,\mathrm{d}\sigma_{\nu} = \oint_{S}\xi^{\mu}_{(n)}\tilde{U}^{\nu\rho}_{\mu}\,\mathrm{d}S_{\nu\rho} \tag{2.10}$$

(the integral conservation law). This expression is invariant with respect to arbitrary space–time transformations.

3. 'Translation–rotation' group in the generalised harmonic frame of reference

In order to derive the actual conservation laws of the four-momentum and the four-angular momentum from the general expressions (2.6)–(2.9), let us consider the 'translation–rotation' group.

Theorem. The infinitesimal 'translation–rotation' group is uniquely defined by

$$\begin{aligned}&\xi^{\mu}_{(n)}=\bar{g}^{\mu\lambda}C_{(n)\lambda}, \qquad n=1,2,3,\ldots,10,\\&\xi_{(n)\mu}=g_{\mu\lambda}\xi^{\lambda}_{(n)}=g_{\mu\lambda}\bar{g}^{\lambda\rho}C_{(n)\rho},\\&\bar{\nabla}_{\mu}C_{(n)\lambda}+\bar{\nabla}_{\lambda}C_{(n)\mu}=0,\end{aligned}\tag{3.1}$$

in the generalised harmonic frames of references, in which the Rosen–Fock conditions (Rosen 1940, 1963) $\bar{\nabla}_{\nu}\sqrt{-g}\,g^{\mu\nu}=0$ are satisfied.

Proof. Let (x, y, z, t) and $('x, 'y, 'z, 't)$ denote harmonically rectangular coordinates and $'x^{\mu}=x^{\mu}+\varepsilon_{(n)}\xi^{\mu}_{(n)}$; then we have (Belinfante 1955, Belinfante and Carrison 1982)

$$\begin{aligned}&\varepsilon_{(n)}\xi^{\mu}_{(n)}=C^{\mu}+w^{\mu}_{\lambda}\chi^{\lambda}=\eta^{\mu\nu}C_{\nu}+\eta^{\mu\nu}w_{\nu\lambda}\chi^{\lambda},\\&C_{\nu}=\text{constant}, \qquad w_{\nu\lambda}=-w_{\lambda\nu}=\text{constant},\\&\eta^{\mu\lambda}=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\end{aligned}\tag{3.2}$$

i.e.

$$\begin{aligned}&\xi^{\mu}_{(n)}=\eta^{\mu\nu}C_{(n)\nu},\\&\partial C_{(n)\lambda}/\partial\chi^{\mu}+\partial C_{(n)\mu}/\partial\chi^{\lambda}=0, \qquad n=1,2,3,\ldots,10.\end{aligned}\tag{3.3}$$

From the covariant condition, we find from (3.3)

$$\begin{aligned}&\xi^{\mu}_{(n)}=\bar{g}^{\mu\nu}C_{(n)\nu},\\&\bar{\nabla}_{\mu}C_{(n)\lambda}+\bar{\nabla}_{\lambda}C_{(n)\mu}=0, \qquad n=1,2,3,\ldots,10.\end{aligned}\tag{3.4}$$

From the theorem above, the conservation laws of four-moentum and four-angular momentum are of the forms

$$\begin{aligned}&\bar{\nabla}_{\nu}\Theta^{\nu}_{(n)}=0,\\&\Theta^{\nu}_{(n)}=\bar{\nabla}_{\rho}((1/\sqrt{-\bar{g}})\xi^{\mu}_{(n)}\tilde{U}^{\nu\rho}_{\mu}),\\&I_{(n)}=(16\pi)^{-1}\oint_{S}\sqrt{-\bar{g}}\,\xi^{\mu}_{(n)}\bar{g}_{\mu\lambda}\tilde{\bar{\nabla}}_{\beta}\mathcal{T}^{\rho\nu\beta\lambda}\,\mathrm{d}S_{\nu\rho},\end{aligned}\tag{3.5}$$

with

$$\begin{aligned}&\xi^{\mu}_{(n)}=\bar{g}^{\mu\lambda}C_{(n)\lambda}, \qquad n=1,2,3,\ldots,10,\\&\xi_{\mu(n)}=g_{\mu\lambda}\xi^{\lambda}_{(n)},\\&\bar{\nabla}_{\mu}C_{(n)\lambda}+\bar{\nabla}_{\lambda}C_{(n)\mu}=0.\end{aligned}\tag{3.6}$$

4. The energy of the spherical bodies

In the harmonically spherical coordinate frame, we have

$$g_{\mu\nu}=\begin{pmatrix} -(r+m)/(r-m) & 0 & 0 & 0 \\ 0 & -(r+m)^2 & 0 & 0 \\ 0 & 0 & -(r+m)^2\sin^2\theta & 0 \\ 0 & 0 & 0 & (r-m)/(r+m) \end{pmatrix} \tag{4.1}$$

and the corresponding flat metric is

$$\bar{g}_{\mu\nu}=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

Inserting (4.1)–(4.2) into the flat Killing equation (3.1), we get

$$\begin{aligned} C_{(0)\mu} &= (0, 0, 0, 1) \\ \xi^{\mu}_{(0)} &= \bar{g}^{\mu\lambda}C_{(0)\lambda} = (0, 0, 0, 1) \end{aligned} \qquad \text{('time-translation')} \tag{4.3}$$

and inserting (4.3) into (3.5) and (2.8), we obtain

$$\begin{aligned} \Theta^{\nu}_{(0)} &= \bar{\nabla}_{\rho}((1/\sqrt{-\bar{g}})\xi^{\mu}_{(0)}\tilde{U}^{\nu\rho}_{\mu}) = \bar{\nabla}_{\rho}((1/\sqrt{-\bar{g}})\tilde{U}^{\nu\rho}_{0}) \\ &= (1/\sqrt{-\bar{g}})(\partial/\partial\chi^{\rho})\tilde{U}^{\nu\rho}_{0} \qquad \text{(four-flux of the energy)}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} E_r &= \oint_{r,t=c} \xi^{0}_{(0)}\tilde{U}^{\nu\rho}_{0}\,\mathrm{d}S_{\nu\rho} \\ &= \oint_{r,t=c} \tilde{U}^{01}_{0}\,\mathrm{d}\theta\,\mathrm{d}\phi \qquad \text{(energy contained in a sphere)}. \end{aligned} \tag{4.5}$$

Furthermore, inserting (4.1)–(4.2) into (2.8), we get

$$\tilde{U}^{01}_{0} = (4\pi)^{-1}(m - m^2/2r)\sin\theta, \tag{4.6}$$

$$\tilde{U}^{0\rho}_{0} = 0 \quad (\rho \neq 1), \qquad \tilde{U}^{\kappa\rho}_{0} = 0 \quad (\kappa = 1, 2, 3). \tag{4.7}$$

Thus, from (4.4)–(4.7), we have

$$\begin{aligned} &\Theta^{0}_{(0)}\ \text{(energy density)} = m^2/8\pi r^4 > 0, \\ &\Theta^{\kappa}_{0}\ \text{(flux of energy)} = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &E_r = m - m^2/2r, \\ &E_g = \lim_{r\to m} E_r = m/2, \\ &E\ \text{(total energy)} = \lim_{r\to\infty} E_r = m. \end{aligned} \tag{4.9}$$

From (4.9) we see that:

(a) The energy density is positive definite.

(b) The gravitational 'self-energy' is $E_s = -m^2/2r$ which is in accordance with the 'Newtonian formula'.

(c) The energy–mass relation obtained is in accordance with the empirical formula (cf table 1).

Table 1. The comparison of the present scheme with others.

Author	Energy computed from the harmonically spherical coordinates frame
Einstein (1916a, b)	$E_r = m - r, E_g = 0, E_s = -r, E = -\infty$
Landau and Lifshitz (1951)	$E_r = -\frac{1}{4}\pi(r+m)^3, E_g = -2\pi m^3, E = -\infty$
Infeld (1960)	$E_r = ((r+m)/(r-m))^2(m - \frac{1}{2}r), E_g = \infty, E = -\infty$
Moller (1959)	$E_r = E_g = E = m, E_s = 0$
Cornish (1965)	$E_r = (m - m^2/2r)[(r+m)/(r-m)](1+m/r)^2, E_g = \infty,$ $E = m$
Author of this paper	$E_r = m - m^2/2r, E_g = \frac{1}{2}m, E_s = -m^2/2r, E = m$
Empirical formula	$E_g \neq \infty, E_s \neq 0, \infty, E = m$

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